The flow induced by a disk oscillating in its own plane

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The flow field due to the oscillations of a disk is governed largely by diffusion near the boundary, but the inertia forces cannot be neglected at large distances. The solution is obtained as a single expansion for the case when the disk is performing small-amplitude sinusoidal oscillations.

1. Introduction

When a body immersed in a fluid of density ρ and kinematic viscosity ν performs small-amplitude tangential oscillations of frequency ω it is well known that the motion is confined to a Stokes boundary layer of thickness $O(\nu/\omega)^{\frac{1}{2}}$. If, however, there is a gradient of centrifugal forces parallel to the boundary it is clear that some steady circulatory motion will be set up in the far field. This motion will be controlled by the viscous-inertia force balance. Thus the first-order motion will be of a different type near to and far away from the body. Problems of this nature can be investigated by finding inner and outer solutions and then using some kind of matching process. In the case of the oscillating disk this type of analysis was done by Rosenblat (1959). An investigation of the problem with special reference to the behaviour of liquid helium has been made by Gribben (1961). The flow outside bodies of revolution oscillating at small amplitude has been treated by Kestin & Persen (1954), and the second-order motions were calculated by Carrier & Di Prima (1956) for the case of a spherical body.

In this paper the discussion will be limited to the configuration considered by Rosenblat. However, the techniques involved in the solution are quite different, and it is believed that a clearer picture of the physics results. Further there appears to be no difficulty in extending the method to obtain the higher approximations and indeed to other geometries. Our primary purpose is to find a single series representation valid over the entire flow field; namely near the disk where the flow must be nearly of the Stokes boundary-layer type, and at infinity where one would anticipate a uniform inflow due to centrifugal action.

2. Equations

The disk is supposed to be of infinite radius and coincident with the plane $z = 0$. The fluid is taken to be homogeneous and incompressible and to lie in the region $z > 0$. Taking cylindrical co-ordinates (r, θ, z) and the corresponding velocity components as (u, v, w), the equations governing the motion are

$$u_r + \frac{u}{r} + \frac{1}{r} v_\theta + w_z = 0, \tag{2.1}$$

$$u_t + uu_r + \frac{v}{r} u_\theta + wu_z - \frac{v^2}{r} = -\frac{1}{\rho} p_r + \nu \left(\Delta u - \frac{2}{r^2} v_\theta - \frac{u}{r^2} \right), \tag{2.2}$$

$$v_t + uv_r + \frac{v}{r} v_\theta + wv_z + \frac{uv}{r} = -\frac{1}{\rho r} p_\theta + \nu \left(\Delta v + \frac{2}{r^2} u_\theta - \frac{v}{r^2} \right), \tag{2.3}$$

$$w_t + uw_r + \frac{v}{r} w_\theta + ww_z = -\frac{1}{\rho} p_z + \nu \Delta w, \tag{2.4}$$

with the boundary conditions

$$u = w = 0, \quad v = \Omega r (e^{i\omega t} + e^{-i\omega t}), \quad \text{at} \quad z = 0; \tag{2.5}$$

$$u, v \to 0 \quad \text{as} \quad z \to \infty. \tag{2.6}$$

The motion is assumed to be independent of θ and so we may introduce a stream function (see Goldstein 1938) and write

$$u = r^{-1} \psi_z, \quad v = r^{-1} \phi, \quad w = -r^{-1} \psi_r. \tag{2.7}$$

The two relevant equations are

$$D^2 \psi_t - \frac{2}{r^2} \phi \phi_z - \frac{1}{r} \frac{\partial(\psi, D^2 \psi)}{\partial(r, z)} - \frac{2}{r^2} \psi_z D^2 \psi = \nu D^4 \psi, \tag{2.8}$$

$$\phi_t - \frac{1}{r} \frac{\partial(\psi, \phi)}{\partial(r, z)} = \nu D^2 \phi, \tag{2.9}$$

where
$$D^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{2.10}$$

By virtue of the boundary conditions we assume that

$$\psi = r^2 F(z, t), \quad \phi = r^2 G(z, t), \tag{2.11}$$

and after one integration of equation (2.8) obtain

$$\nu F_{zzz} - F_{zt} = -G^2 + F_z^2 - 2FF_{zz}, \tag{2.12}$$

$$\nu G_{zz} - G_t = 2(F_z G - F G_z). \tag{2.13}$$

On introducing dimensionless variables (denoted by primes)

$$z = (2\nu/\omega)^{\frac{1}{2}} z', \quad t = (1/\omega) t', \quad F = \Omega (2\nu/\omega)^{\frac{1}{2}} F', \quad G = \Omega G', \tag{2.14}$$

the problem reduces to that of solving

$$\tfrac{1}{2} F_{zzz} - F_{zt} = \epsilon [-G^2 + F_z^2 - 2FF_{zz}], \tag{2.15}$$

$$\tfrac{1}{2} G_{zz} - G_t = 2\epsilon [F_z G - F G_z], \tag{2.16}$$

subject to the boundary conditions,

$$F(0, t) = F_z(0, t) = 0, \quad G(0, t) = e^{it} + e^{-it}; \tag{2.17}$$

$$F_z(\infty, t) = G(\infty, t) = 0, \tag{2.18}$$

where $\epsilon = \Omega/\omega$ and the primes are now omitted. These are the equations considered by Rosenblat.

In this paper we are interested in the case ϵ small and positive; that is when the thickness of the Stokes layer $(2\nu/\omega)^{\frac{1}{2}}$ is very much less than that of the Ekman layer $(2\nu/\Omega)^{\frac{1}{2}}$. If we considered the case $\epsilon \gg 1$ it would be natural to use the Ekman layer as the length scale and the equations would reduce to

$$\tfrac{1}{2}F_{zzz} - \frac{1}{\epsilon}F_{zt} = -G^2 + F_z^2 - 2FF_{zz}, \tag{2.19}$$

$$\tfrac{1}{2}G_{zz} - \frac{1}{\epsilon}G_t = 2(F_zG - FG_z). \tag{2.20}$$

With $\epsilon^{-1} = 0$ these are the equations for the steadily rotating disk of von Kármán (1921).

3. Method of solution

Consider now the equations (2.15), (2.16) with the boundary conditions (2.17), (2.18) and ϵ small and positive. As has been mentioned earlier the basic difficulty with a straightforward Taylor expansion of F and G in powers of ϵ is that the far field involves a non-linear balance in the equations, and so such series can only represent the true solution within the Stokes layer. This difficulty soon becomes apparent in the analysis of the higher perturbations where powers of z arise in the solution. In anticipation of these complications it is convenient to introduce two length scales into the problem at the outset, and to solve for the velocity components considered as functions of three independent variables z, t, and $\xi = \epsilon z$. The ξ-dependence will be chosen to suppress any difficulties in the perturbation procedure. This idea appears to originate from the work of Mahony (1961) and has been used by Kevorkian (1961) in problems involving non-linear oscillations. This device is artificial; but, at least for the present problem, is most convenient.

Introducing $\xi = \epsilon z$ as an additional independent variable, the problem is to find $F(z, \xi, t)$ and $G(z, \xi, t)$ where

$$\tfrac{1}{2}F_{zzz} - F_{zt} = \epsilon[-\tfrac{3}{2}F_{zz\xi} + F_{\xi t} - G^2 + F_z^2 - 2FF_{zz}] + \epsilon^2[-\tfrac{3}{2}F_{z\xi\xi} + 2F_zF_\xi - 4FF_{z\xi}] + \epsilon^3[-\tfrac{1}{2}F_{\xi\xi\xi} + F_\xi^2 - 2FF_{\xi\xi}], \tag{3.1}$$

$$\tfrac{1}{2}G_{zz} - G_t = \epsilon[-G_{z\xi} + 2F_zG - 2FG_z] + \epsilon^2[-\tfrac{1}{2}G_{\xi\xi} + 2F_\xi G - 2FG_\xi], \tag{3.2}$$

$$F(0,0,t) = F_z(0,0,t) + \epsilon F_\xi(0,0,t) = 0, \quad G(0,0,t) = e^{it} + e^{-it}; \tag{3.3}$$

$$F_z(\infty,\infty,t) + \epsilon F_\xi(\infty,\infty,t) = G(\infty,\infty,t) = 0. \tag{3.4}$$

We assume
$$F = \sum_{n=0}^{\infty} \epsilon^n F^{(n)}(z,\xi,t), \tag{3.5}$$

$$G = \sum_{n=0}^{\infty} \epsilon^n G^{(n)}(z,\xi,t). \tag{3.6}$$

For the zeroth-order approximation we have

$$\tfrac{1}{2}F^{(0)}_{zzz} - F^{(0)}_{zt} = 0, \tag{3.7}$$

$$\tfrac{1}{2}G^{(0)}_{zz} - G^{(0)}_t = 0. \tag{3.8}$$

In anticipation of the expected inflow at infinity and the boundary conditions on G we take

$$F^{(0)} = f_0^{(0)}(z,\xi), \tag{3.9}$$

$$G^{(0)} = g_1^{(0)}(z,\xi)\,e^{it} + g_1^{(0)*}(z,\xi)\,e^{-it}, \tag{3.10}$$

where

$$f_0^{(0)} = a_{00}^{(0)} + a_{01}^{(0)}z + a_{02}^{(0)}z^2, \tag{3.11}$$

$$g_1^{(0)} = b_{11}^{(0)}\,e^{-(1+i)z} + b_{12}^{(0)}\,e^{(1+i)z}. \tag{3.12}$$

The $a_{ij}^{(k)}$ and $b_{ij}^{(k)}$ are as yet arbitrary functions of ξ. In order to have a uniformly valid expansion we must insist on the dominance of the nth-order terms over the $(n+1)$st order terms. Thus certain particular integrals must be avoided by choosing an appropriate ξ depencence for the solutions. For the moment we leave aside any question of boundary conditions.

The equations for the first approximation are

$$\tfrac{1}{2}F_{zzz}^{(1)} - F_{zt}^{(1)} = -\tfrac{3}{2}F_{zz\xi}^{(0)} + F_{\xi t}^{(0)} - G^{(0)2} + F_z^{(0)2} - 2F^{(0)}F_{zz}^{(0)}, \tag{3.13}$$

$$\tfrac{1}{2}G_{zz}^{(1)} - G_t^{(1)} = -G_{z\xi}^{(0)} + 2(F_z^{(0)}G^{(0)} - F^{(0)}G_z^{(0)}), \tag{3.14}$$

and so

$$F^{(1)} = f_0^{(1)}(z,\xi) + f_2^{(1)}(z,\xi)\,e^{2it} + f_2^{(1)*}(z,\xi)\,e^{-2it}, \tag{3.15}$$

$$G^{(1)} = g_1^{(1)}(z,\xi)\,e^{it} + g_1^{(1)*}(z,\xi)\,e^{-it}. \tag{3.16}$$

Clearly positive exponentials must be absent else these would propagate into the higher approximations with larger exponents. Thus, for example,

$$b_{12}^{(0)} = 0. \tag{3.17}$$

For $g_1^{(1)}$ we have

$$\tfrac{1}{2}g_{1zz}^{(1)} - ig_1^{(1)} = e^{-(1+i)z}[(1+i)\{b_{11}^{(0)\prime} + 2(a_{00}^{(0)} + a_{01}^{(0)}z + a_{02}^{(0)}z^2)\,b_{11}^{(0)}\} + 2\{a_{01}^{(0)} + 2a_{02}^{(0)}z\}\,b_{11}^{(0)}]. \tag{3.18}$$

Any term of the form $z^n e^{-(1+i)z}$ $(n \geqslant 0)$ on the right-hand side of equation (3.18) gives rise to a term of the form $z^{n+1}e^{-(1+i)z}$ in $g_1^{(1)}$ and so would be inadmissible. Therefore we insist that

$$a_{01}^{(0)} = a_{02}^{(0)} = 0, \tag{3.19}$$

$$b_{11}^{(0)\prime} + 2a_{00}^{(0)}\,b_{11}^{(0)} = 0, \tag{3.20}$$

where the primes denote differentiations with respect to ξ.

Similar considerations apply to the equations for $f_0^{(1)}$ and $f_2^{(1)}$,

$$\tfrac{1}{2}f_{0zzz}^{(1)} = -2b_{11}^{(0)}\,b_{11}^{(0)*}\,e^{-2z}, \tag{3.21}$$

$$\tfrac{1}{2}f_{2zzz}^{(1)} - 2if_{2zt}^{(1)} = -b_{11}^{(0)2}\,e^{-2(1+i)z}. \tag{3.22}$$

The most general admissible solutions are

$$f_0^{(1)} = a_{00}^{(1)} + \tfrac{1}{2}b_{11}^{(0)}\,b_{11}^{(0)*}\,e^{-2z}, \tag{3.23}$$

$$f_2^{(1)} = a_{20}^{(1)} + a_{21}^{(1)}\,e^{-\sqrt{2}(1+i)z} - \tfrac{1}{8}(1+i)\,b_{11}^{(0)2}\,e^{-2(1+i)z}, \tag{3.24}$$

$$g_1^{(1)} = b_{11}^{(1)}\,e^{-(1+i)z}. \tag{3.25}$$

The continuation of this process presents only algebraic difficulties. For the present purpose we shall content ourselves with finding the uniform expansion correct to the first order. This involves a determination of the functions $a_{00}^{(0)}(\xi)$, $b_{11}^{(0)}(\xi)$, $a_{00}^{(1)}(\xi)$, $a_{20}^{(1)}(\xi)$, $a_{21}^{(1)}(\xi)$, $b_{11}^{(1)}(\xi)$ and necessitates considering equations up to the fourth order.

For $F^{(2)}$ and $G^{(2)}$ we have

$$\tfrac{1}{2}F^{(2)}_{zzz}-F^{(2)}_{zt} = -\tfrac{3}{2}F^{(1)}_{zz\xi}+F^{(1)}_{\xi t}-2G^{(0)}G^{(1)}+2F^{(0)}_{z}F^{(1)}_{z}-2F^{(0)}F^{(1)}_{zz}-2F^{(1)}F^{(0)}_{zz}$$
$$-\tfrac{3}{2}F^{(0)}_{z\xi\xi}+2F^{(0)}_{z}F^{(0)}_{\xi}-4F^{(0)}F^{(0)}_{z\xi}, \tag{3.26}$$
$$\tfrac{1}{2}G^{(2)}_{zz}-G^{(2)}_{t} = -G^{(1)}_{z\xi}+2F^{(0)}_{z}G^{(1)}+2F^{(1)}_{z}G^{(0)}-2F^{(0)}G^{(1)}_{z}-2F^{(1)}G^{(0)}_{z}$$
$$-\tfrac{1}{2}G^{(0)}_{\xi\xi}+2F^{(0)}_{\xi}G^{(0)}-2F^{(0)}G^{(0)}_{\xi}, \tag{3.27}$$

and so
$$F^{(2)} = f_0^{(2)}(z,\xi)+f_2^{(2)}(z,\xi)\,e^{2it}+f_2^{(2)*}(z,\xi)\,e^{-2it}, \tag{3.28}$$
$$G^{(2)} = g_1^{(2)}(z,\xi)\,e^{it}+g_1^{(2)*}(z,\xi)\,e^{-it}+g_3^{(2)}(z,\xi)\,e^{3it}+g_3^{(2)*}(z,\xi)\,e^{-3it}. \tag{3.29}$$

In the equation for $g_1^{(2)}$ we must have the coefficient of $e^{-(1+i)z}$ on the right-hand side as zero. This yields the equation

$$b_{11}^{(1)\prime}+2a_{00}^{(0)}b_{11}^{(1)}+2a_{00}^{(1)}b_{11}^{(0)}+(1-i)\,a_{00}^{(0)2}b_{11}^{(0)}+\tfrac{3}{2}(1-i)\,a_{00}^{(0)\prime}b_{11}^{(0)} = 0. \tag{3.30}$$

The equations for $f_0^{(2)}$ and $g_3^{(2)}$ present no difficulties; but for $f_2^{(2)}$ we require

$$a_{20}^{(1)\prime} = 0, \tag{3.31}$$
$$a_{21}^{(1)\prime}+2a_{00}^{(0)}a_{21}^{(1)} = 0. \tag{3.32}$$

At each stage the ξ dependence becomes determined to a certain order. To obtain the solution correct to the first order it suffices to consider the equations for $f_0^{(3)}$ and $f_0^{(4)}$ in $F^{(3)}$ and $F^{(4)}$ to close the system. These require

$$a_{00}^{(0)\prime\prime\prime}-2a_{00}^{(0)\prime 2}+4a_{00}^{(0)}a_{00}^{(0)\prime\prime} = 0, \tag{3.33}$$
$$a_{00}^{(1)\prime\prime\prime}+4a_{00}^{(0)}a_{00}^{(1)\prime\prime}-4a_{00}^{(0)\prime}a_{00}^{(1)\prime}+4a_{00}^{(0)\prime\prime}a_{00}^{(1)} = 0. \tag{3.34}$$

The boundary conditions can be found using equations (3.3) and (3.4). We find the following sequence of problems for $a_{ij}^{(k)}$, $b_{ij}^{(k)}$;

$$\left.\begin{aligned} a_{00}^{(0)\prime\prime\prime} &= 2a_{00}^{(0)\prime 2}-4a_{00}^{(0)}a_{00}^{(0)\prime\prime},\\ a_{00}^{(0)}(0) = 0, \quad a_{00}^{(0)\prime}(0) &= 1, \quad a_{00}^{(0)\prime}(\infty) = 0;\end{aligned}\right\} \tag{3.35}$$

$$\left.\begin{aligned} b_{11}^{(0)\prime}+2a_{00}^{(0)}b_{11}^{(0)} &= 0,\\ b_{11}^{(0)}(0) &= 1;\end{aligned}\right\} \tag{3.36}$$

$$\left.\begin{aligned} a_{20}^{(1)\prime} &= 0,\\ a_{20}^{(1)}(0) &= \frac{(1-\sqrt{2})(1+i)}{8};\end{aligned}\right\} \tag{3.37}$$

$$\left.\begin{aligned} a_{21}^{(1)\prime}+2a_{00}^{(0)}a_{21}^{(1)} &= 0,\\ a_{21}^{(1)}(0) &= \frac{1+i}{4\sqrt{2}};\end{aligned}\right\} \tag{3.38}$$

$$\left.\begin{aligned} a_{00}^{(1)\prime\prime\prime}+4a_{00}^{(0)}a_{00}^{(1)\prime\prime}-4a_{00}^{(0)\prime}a_{00}^{(1)\prime}+4a_{00}^{(0)\prime\prime}a_{00}^{(1)} &= 0,\\ a_{00}^{(1)}(0) = -\tfrac{1}{2}, \quad a_{00}^{(1)\prime}(0) = 0, \quad a_{00}^{(1)\prime}(\infty) &= 0;\end{aligned}\right\} \tag{3.39}$$

$$\left.\begin{aligned} b_{11}^{(1)\prime}+2a_{00}^{(0)}b_{11}^{(1)}+2a_{00}^{(1)}b_{11}^{(0)}+(1-i)\,a_{00}^{(0)2}b_{11}^{(0)}+\tfrac{3}{2}(1-i)\,a_{00}^{(0)\prime}b_{11}^{(0)} &= 0,\\ b_{11}^{(1)}(0) &= 0.\end{aligned}\right\} \tag{3.40}$$

4. Concluding remarks

The equations (3.35)–(3.40) are a set of ordinary differential equations capable of being solved numerically or by approximate methods. Clearly $a_{00}^{(0)}$ and $a_{00}^{(1)}$ will tend exponentially to certain constants as $\xi \to \infty$. The function $a_{20}^{(1)}$ is constant everywhere and $b_{11}^{(0)}$, $a_{21}^{(1)}$ and $b_{11}^{(1)}$ will decay exponentially to zero in ξ. These latter three functions of ξ multiply functions which are also decaying in z. Also as $a_{20}^{(1)}$ is constant the time-dependent part of the motion which is dominant at infinity is an oscillatory suction velocity having twice the frequency of the forced motion.

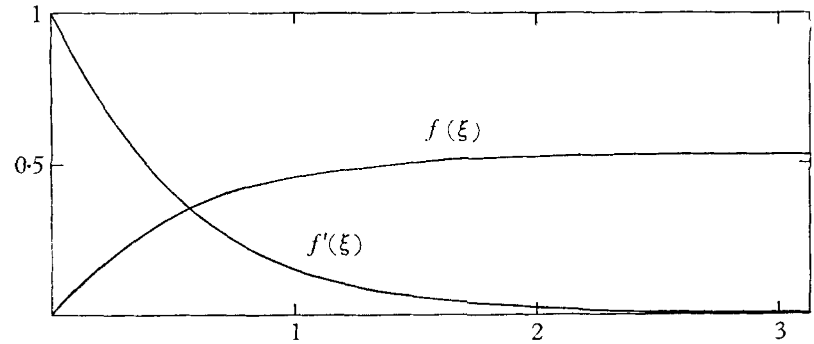

FIGURE 1. Graphs of $f(\xi)$, $f'(\xi)$.

The dimensional mean velocities (denoted by u_m and w_m) are

$$u_m = \epsilon\Omega r\{a_{00}^{(0)\prime} - b_{11}^{(0)} b_{11}^{(0)*} e^{-2z} + O(\epsilon)\}, \tag{4.1}$$

$$w_n = -2\Omega(2\nu/\omega)^{\frac{1}{2}}\{a_{00}^{(0)} + O(\epsilon)\}. \tag{4.2}$$

It is of interest to determine $a_{00}^{(0)}(\infty)$, the mean inflow velocity, and to note that the mean radial flow peaks at the outer edge of the Stokes layer, its magnitude being independent of viscosity.

The usual approximate methods are available for dealing with the above set of equations. We take one simple iterative process for finding $a_{00}^{(0)}$ (which we now denote by f for convenience) which compares favourably with an exact numerical integration. This iteration is

$$f'''_{n+1} = 2f'^2_n - 4f_n f''_n, \tag{4.3}$$

with

$$f_0 = (1-e^{-\lambda\xi})/\lambda. \tag{4.4}$$

Successive values of λ are to be determined to make the iterates satisfy the boundary conditions. We find that

$$f_1 = [4(1-e^{-\lambda\xi}) - \tfrac{1}{4}(1-e^{-2\lambda\xi})]/\lambda^3, \tag{4.5}$$

$$\lambda_1 = 1{\cdot}87. \tag{4.6}$$

We find $f_1(\infty) = 0{\cdot}573$, $f_1''(0) = -1{\cdot}60$; and a further iteration gives $f_2(\infty) = 0{\cdot}547$, $f_2''(0) = -1{\cdot}64$. These results show reasonable agreement with the solution obtained by a direct numerical integration, $f(\infty) = 0{\cdot}530$, $f''(0) = -1{\cdot}66$. Figure 1 shows the graphs of $f(\zeta)$ and $f'(\zeta)$.

Finally, the shear stress τ at the plate is found to be identical to that given by Rosenblat, namely

$$\tau = 2\Omega\rho\nu(\omega/2\nu)^{\frac{1}{2}} r[(-1+0{\cdot}761\epsilon^2)\cos 3\omega t + (1-1{\cdot}046\epsilon^2)\sin\omega t + 0{\cdot}041\epsilon^2(\cos 3\omega t - \sin 3\omega t) + \ldots]. \quad (4.7)$$

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